On Indecomposable Normal Matrices in Spaces with Indefinite Scalar Product

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Abstract

Finite dimensional linear spaces (both complex and real) with indefinite scalar product $[\cdot,\cdot]$ are considered. Upper and lower bounds are given for the size of an indecomposable matrix that is normal with respect to this scalar product in terms of specific functions of $v = \min\{v_-, v_+\}$, where v_- (v_+) is the number of negative (positive) squares of the form $[x, x]$. All the bounds except for one are proved to be strict.

1 Definitions and notation

Consider a complex (real) linear space C^n (R^n) with an indefinite scalar product $[\cdot,\cdot]$. By definition, the latter is a nondegenerate sesquilinear (bilinear) Hermitian form. If the usual scalar product $(\cdot,\cdot)$ is fixed, then there exists a nonsingular Hermitian operator H such that $[x, y] = (Hx, y)$ $\forall x, y \in C^n$ (R^n). If A is a linear operator, then the *H-adjoint* of A (denoted by $A^{[*]}$) is defined by the identity $[A^{[*]}x, y] \equiv [x, Ay]$. An operator N is called *H-normal* if $NN^{[*]} = N^{[*]}N$. An operator U is called *H-unitary* if $UU^{[*]} = I$, where I is the identity transformation.

Let V be a nontrivial subspace of C^n (R^n). The subspace V is called *neutral* if $[x, y] = 0$ $\forall x, y \in V$. If the conditions $x \in V$ and $[x, y] = 0$ $\forall y \in V$ imply $x = 0$, then V is called *nondegenerate*. The subspace $V^{[\perp]}$ is defined as the set of all vectors x from C^n (R^n) such that $[x, y] = 0$ $\forall y \in V$. If V is nondegenerate, then $V^{[\perp]}$ is also nondegenerate and $V \dot{+} V^{[\perp]} = C^n$ (R^n), where $\dot{+}$ stands for the direct sum.

A linear operator A is called *decomposable* if there exists a nondegenerate proper subspace V of C^n (R^n) such that both V and $V^{[\perp]}$ are invariant under A or (it is the same) if V is invariant both under A and $A^{[*]}$. Then A is the *H-orthogonal sum of* $A_1 = A|_V$ *and* $A_2 = A|_{V^{[\perp]}}$. If an operator A is not decomposable, it is called *indecomposable*.

By the *rank* of a space we mean $v = \min\{v_-, v_+\}$, where v_- (v_+) is the number of negative (positive) squares of the form $[x, x]$, i.e., the number of negative (positive) eigenvalues of the operator H.

The problem is to find functions $f_1(\cdot)$, $f_2(\cdot)$ such that $f_1(v) \leq n \leq f_2(v)$ for any indecomposable H-normal operator acting in a space of dimension n and of rank v and to find out whether these bounds are strict.

This problem arises in the classification of indecomposable H-normal matrices [2, 3]. The bounds for the size of an indecomposable H-normal matrix in a complex space are known [2]. In Section 2, we check their strictness. The bounds for matrices in real spaces are considered in Section 3.

As in [2] and [3], we denote by I_r the $r \times r$ identity matrix, by D_r the $r \times r$ matrix with 1's on the trailing diagonal and zeros elsewhere, and by $A \oplus B \oplus \ldots \oplus Z$ the block diagonal matrix with blocks A, B, …, Z. By A^T we mean A transposed.

2 Indecomposable normal matrices in complex spaces

The objective of this section is to prove the following theorem.

Theorem 1 *Let an indecomposable H-normal operator N act in a space C^n of rank $k > 0$. Then either (A) or (B) holds:*

(A) N has only one eigenvalue and $2k \le n \le 4k$;

(B) N has only two eigenvalues and $n = 2k$;

these bounds being strict.

Proof: Theorem 1 of [2] states that for an indecomposable H-normal operator N there exist two alternatives: (A) and (B) so that it suffices to prove that these estimates are unimprovable.

Step 1. Show the strictness of the low bound in (A), i.e., for any $k > 0$ point out a pair of $2k \times 2k$ matrices $\{N, H\}$, where H has k negative and k positive eigenvalues, N is H-normal and indecomposable and has only one eigenvalue λ. Let

$$N = \begin{pmatrix} \lambda I_k & N_1 \\ 0 & \lambda I_k \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix}. \tag{1}$$

It can easily be checked that N is H-normal. In addition, let the submatrix N_1 be nonsingular.

Proposition 1 from [2] and its Corollary may be restated as follows:

Let an H-normal operator N acting in C^n have λ as its only eigenvalue. Let the subspace

$$S_0 = \{x \in C^n : \ (N - \lambda I)x = (N^{[*]} - \overline{\lambda} I)x = 0\} \tag{2}$$

be neutral. Then there exists a decomposition of C^n into a direct sum of subspaces S_0, S, S_1 such that

$$N = \begin{pmatrix} N' = \lambda I & * & * \\ 0 & N_1 & * \\ 0 & 0 & N'' = \lambda I \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & 0 & I \\ 0 & H_1 & 0 \\ I & 0 & 0 \end{pmatrix}, \tag{3}$$

where $N' : S_0 \to S_0$, $N_1 : S \to S$, $N'' : S_1 \to S_1$, the internal operator N_1 is H_1-normal, and the pair $\{N_1, H_1\}$ is determined up to the unitary similarity. To go over from one decomposition $C^n = S_0 \dot{+} S \dot{+} S_1$ to another by means of a transformation T it is necessary that T be block triangular with respect to both decompositions.

(In Proposition 1 from [2] there are two conditions:

(a) N is indecomposable

(b) $n > 1$

instead of the condition

(c) S_0 is neutral,

but the results are only derived from (c), which follows from (a) and (b)).

We see that (1) is a specific case of (3) corresponding to the decomposition $C^n = S_0 \dot{+} S \dot{+} S_1$ with $S = 0$. If there exists a nondegenerate subspace V such that both $V_1 = V$ and $V_2 = V^{[\perp]}$ are invariant under N, then, according to our restatement of Proposition 1 from [2], for $i = 1, 2$ we have $V^{(i)} = S_0^{(i)} \dot{+} S^{(i)} \dot{+} S_1^{(i)}$, where $S_0^{(i)} = V^{(i)} \cap S_0$ and the pairs $\{N^{(i)}, H^{(i)}\}$ have the form (3). But (1) implies $S_0 = S_0^{[\perp]}$ so that for any $i = 1, 2$ the subspace $S^{(i)}$ is trivial. Thus, N from (1) is decomposable if and only if there exists a transformation T preserving H and reducing N to the form

$$\widetilde{N} = \begin{pmatrix} \lambda I_k & \widetilde{N_1} \\ 0 & \lambda I_k \end{pmatrix},$$

where $\widetilde{N_1}$ is block diagonal (that is, T is an H-unitary transformation of N to the form $\widetilde{N}$). The matrix T is necessarily block triangular with respect to the decomposition $C^n = S_0 \dot{+} S_1$, i.e.,

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$

For T to be H-unitary it is necessary to have $T_3 = T_1^{*-1}$. Then from the condition $NT = T\widetilde{N}$ it follows that $N_1 = T_1 \widetilde{N_1} T_1^*$. Therefore, N will be indecomposable if N_1 is not congruent to any block diagonal matrix $\widetilde{N_1}$.

If N_1 and a block diagonal matrix $\widetilde{N_1}$ are congruent, then $N_1 N_1^{*-1}$ is similar to $\widetilde{N_1}\widetilde{N_1}^{*-1}$. Since the latter is also block diagonal, for N_1 to be not congruent to $\widetilde{N_1}$ it is sufficient that $N_1 N_1^{*-1}$ cannot be reduced to block diagonal form.

Let us prove that for any $n = 1, 2, \ldots$ there exists a nonsingular real $(2n-1) \times (2n-1)$ matrix N_1 such that

$$N_1 N_1^{*-1} = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \tag{4}$$

and a nonsingular real $2n \times 2n$ matrix N_1 such that

$$N_1 N_1^{*-1} = \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & -1 \end{pmatrix} \tag{5}$$

(now it is not necessary that N_1 be real, but this will be used in the next section). The matrices (4) and (5) are obviously not similar to any block diagonal ones because for each of them the subspace generated by their eigenvectors is one-dimensional.

Prove the statement by induction for odd numbers. If $n = 1$, let $N_1^{(1)} = (1)$. Suppose we have found a nonsingular real $(2n-1) \times (2n-1)$ matrix $N_1^{(n)}$ with the property required. Let

$$N_1^{(n+1)} = \begin{pmatrix} 0 & A_{n+1} & B_{n+1} \\ C_{n+1} & N_1^{(n)} & 0 \\ D_{n+1} & 0 & 0 \end{pmatrix},$$

where the submatrices A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1} will be shortly specified. If we denote by Λ_n the $(2n-1) \times (2n-1)$ matrix (4), by Λ'_n the $1 \times (2n-1)$ matrix

$$\Lambda'_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and by Λ''_n the $(2n-1) \times 1$ matrix

$$\Lambda''_n = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \end{pmatrix}^T,$$

then the condition $N_1^{(n+1)} = \Lambda_{n+1} N_1^{(n+1)*}$ may be rewritten as follows:

$$\begin{aligned} 0 &= \Lambda'_n A^*_{n+1}, && (6) \\ A_{n+1} &= C^*_{n+1} + \Lambda'_n N_1^{(n)*}, && (7) \\ B_{n+1} &= D^*_{n+1}, && (8) \\ C_{n+1} &= \Lambda_n A^*_{n+1} + \Lambda''_n B^*_{n+1} && (9) \end{aligned}$$

(by the inductive hypothesis, $N_1^{(n)} = \Lambda_n N_1^{(n)*}$). Taking

$$A_{n+1} = \left(\begin{array}{ccccc} 0 & a_2 & a_3 & \dots & a_{2n-1} \end{array} \right), \qquad B_{n+1} = (b),$$
$$a_2,\ a_3,\ \dots,\ a_{2n-1},\ b \in \Re,$$

$C_{n+1} = \Lambda_n A_{n+1}^* + \Lambda_n'' B_{n+1}^*$, $D_{n+1} = B_{n+1}^*$, one can satisfy the conditions (6), (8), (9). Substituting the expression for C_{n+1} in (7), we get the only condition to be satisfied:

$$-N_1^{(n)} \Lambda_n'^* = (\Lambda_n - I) A_{n+1}^* + \Lambda_n'' B_{n+1}^*. \tag{10}$$

Since its right hand side is equal to

$$\left(\begin{array}{ccccc} a_2 & a_3 & \cdots & a_{2n-1} & b \end{array} \right)^T,$$

it always can be satisfied by choosing the appropriate values of a_2, a_3, $\dots$, a_{2n-1}, b. By construction, the elements of each matrix $N_1^{(n)}$ below the trailing diagonal are zeros and those on the trailing diagonal are equal to ± 1 so that $N_1^{(n)}$ is nonsingular. Thus, for odd numbers our statement is proved.

To prove it for even numbers one can take

$$N_1^{(1)} = \left(\begin{array}{cc} \frac{1}{2} & 1 \\ -1 & 0 \end{array} \right)$$

and construct $N_1^{(n+1)}$ from $N_1^{(n)}$ in the same way as before (the details are left to the reader). Step 1 is completed.

Step 2. Show the strictness of the upper bound in (A). The example of the pair $\{N, H\}$ is

$$N = \left(\begin{array}{cccc} \lambda I_k & I_k & 0 & 0 \\ 0 & \lambda I_k & 0 & N_1 \\ 0 & 0 & \lambda I_k & N_2 \\ 0 & 0 & 0 & \lambda I_k \end{array} \right), \quad H = \left(\begin{array}{cccc} 0 & 0 & 0 & I_k \\ 0 & I_k & 0 & 0 \\ 0 & 0 & I_k & 0 \\ I_k & 0 & 0 & 0 \end{array} \right), \tag{11}$$

where

$$N_1 = \left(\begin{array}{ccccc} 0 & r_1 & 0 & \dots & 0 \\ 0 & 0 & r_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & r_{k-1} \\ r_k & 0 & 0 & \dots & 0 \end{array} \right),$$

$$N_2 = \left(\begin{array}{cccc} \sqrt{1-r_1^2} & 0 & \dots & 0 \\ 0 & \sqrt{1-r_2^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{1-r_k^2} \end{array} \right),$$

$r_i \in (0,1)$ $\forall i = 1, \dots, k$, and $r_i \neq r_j$ if $i \neq j$. The matrix H has k negative and $3k$ positive eigenvalues. The matrix N is H-normal, since the condition $N_1^* N_1 + N_2^* N_2 = I$ is satisfied. As before, we see that (11) is a specific case of (3). Suppose a nondegenerate subspace V is invariant both under N and under $N^{[*]}$. Denote the basis vectors of S_0 by $\{v_i\}_{i=1}^k$, the basis vectors of S_1 by $\{w_i\}_{i=1}^k$ (here the basis corresponds to (11)). Let $\widetilde{V}$ be the range of the projection of V onto S_1 along $S_0^{[\perp]}$. It is a subspace of dimension $m > 0$, since V necessarily contains at least one nontrivial vector from S_0 and, therefore, at least one vector with nontrivial projection onto S_1 (otherwise V would be degenerate). Let $\{\sum_{j=1}^k \alpha_{ij} w_j\}_{i=1}^m$ be a basis of $\widetilde{V}$. From the condition $(N - \lambda I)(N^{[*]} - \overline{\lambda} I)V \subseteq V$ it follows that $\{\sum_{j=1}^k \alpha_{ij} v_j\}_{i=1}^m \subset V$. If V is nondegenerate, $\widetilde{V}$ and $\widetilde{S_0} = S_0 \cap V$ are necessarily of the same dimension. Therefore, $\{\sum_{j=1}^k \alpha_{ij} v_j\}_{i=1}^m$ is a basis of $\widetilde{S_0}$.

As $(N-\lambda I)^2V \subseteq V$, $(N^{[*]}-\overline{\lambda}I)^2V \subseteq V$, we obtain $\{\sum_{j=1}^{k}\alpha_{ij}N_1v_j\}_{i=1}^{m} \subset V$, $\{\sum_{j=1}^{k}\alpha_{ij}N_1^*v_j\}_{i=1}^{m} \subset V$. As $V^{[\perp]} \cap S_0 \neq \{0\}$, we have $m(=dim\widetilde{S_0}) < k$. Thus, for N to be decomposable it is necessary that the subspace $\widetilde{S_0}$, which is of dimension more than zero and less than k, be invariant under N_1 and under N_1^*. This means the existence of an orthogonal projection $P(\neq 0,\ I)$ commuting with N_1. But it can easily be checked by direct calculation that from the conditions $N_1P = PN_1$ and $P = P^*$ it follows that $P = \mu I$. Since $P^2 = P$, we have $\mu = 0$ or $\mu = 1$ so that $P = 0$ or $P = I$. The contradiction obtained shows that N is indecomposable. Step 2 is completed.

Step 3. Now for any $k > 0$ let us point out a pair of $2k \times 2k$ matrices $\{N, H\}$, where H has k negative and k positive eigenvalues, N is H-normal and indecomposable and has the two eigenvalues $\lambda_1,\ \lambda_2$ $(\lambda_1 \neq \lambda_2)$. Let

$$N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix}, \tag{12}$$

where the $k \times k$ matrices N_i $(i = 1, 2)$ are as follows:

$$N_1 = \begin{pmatrix} \lambda_1 & 1 & 0 & \dots & 0 \\ 0 & \lambda_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \dots & \lambda_1 \end{pmatrix}, \quad N_2 = \lambda_2 I_k.$$

The matrix N is H-normal, for it satisfies the condition $N_1N_2^* = N_2^*N_1$. Suppose that N is similar to

$$\widetilde{N} = \begin{pmatrix} N|_V = \widetilde{N_1} & 0 \\ 0 & N|_{V^{[\perp]}} = \widetilde{N_2} \end{pmatrix},$$

where V is a nondegenerate subspace. Since the subspace generated by the eigenvectors of N corresponding to the eigenvalue λ_1 is one-dimensional, one of the submatrices $\widetilde{N_1}$, $\widetilde{N_2}$ (for example, $\widetilde{N_2}$) has λ_2 as its only eigenvalue, hence $\widetilde{N_2} = \lambda_2 I$. But any subspace generated by eigenvectors corresponding to the eigenvalue λ_2 is neutral so that V cannot be nondegenerate. This contradiction shows the indecomposability of N. Step 3 is completed. The theorem is proved. □

3 Indecomposable normal matrices in real spaces

The objective of this section is to prove the following theorem.

Theorem 2 *Let an indecomposable H-normal operator N act in a space R^n of rank $k > 0$. Then one of the conditions (A) - (E) holds:*

(A) N has only one real eigenvalue and $2k \le n \le 4k$;

(B) N has only two real eigenvalues and $n = 2k$;

(C) N has only two complex conjugate eigenvalues and $n = 2$ if $k = 1$ and $2k \le n \le 10[k/2] - 2$ if $k > 1$;

(D) N has only one real and one pair of complex conjugate eigenvalues and $n = 2k$;

(E) N has only two pairs of complex conjugate eigenvalues and $n = 2k$.

The alternatives (D) and (E) are possible only if k is even. The estimates (A), (B), (D), (E), and the low bound in (C) are strict.

Proof: That an indecomposable H-normal matrix has one of the five sets of eigenvalues is proved in [3, Lemma 1]. Bounds (A) and (B) are proved in [2, Theorem 1], their strictness in Theorem 1 from the previous section (since the matrices constructed in Theorem 1 are real and any matrix that is indecomposable

in a complex space is also indecomposable in a real one). The condition $n \geq 2k$ is obvious. Indeed, since $k = \min\{v_-, v_+\}$ and $n = v_- + v_+$, we have $n \geq 2k$. Thus, we must consider the cases (C) - (E) only, keeping in mind that $n \geq 2k$.

Step 1. Consider the case (C). Let N have the two distinct eigenvalues $\lambda = \alpha + i\beta$ and $\overline{\lambda} = \alpha - i\beta$. The equality $n = 2$ for $k = 1$ is proved in [3, Theorem 1]. In case when $k = 2$ Theorem 2 of [3] states that $n \leq 8$. So, it remains to prove the inequality $n \leq 10[k/2] - 2$ for $k \geq 3$. To this end recall Proposition 2 from [3]:

Let an indecomposable H-normal operator N acting in R^n $(n > 2)$ have the two distinct eigenvalues $\lambda = \alpha + i\beta$, $\overline{\lambda} = \alpha - i\beta$. Let

$$S_0' = \{z = x + iy \ (x, y \in R^n) : \ Nz = \lambda z, \ N^{[*]}z = \overline{\lambda} z\},$$

$$S_0'' = \{z = x + iy \ (x, y \in R^n) : \ Nz = \lambda z, \ N^{[*]}z = \lambda z\},$$

$\{z_j\}_1^p$ $(\{z_j\}_{p+1}^{p+q})$ be a basis of S_0' (S_0''), and

$$S_0 = \sum_{j=1}^{p+q} span\{x_j, y_j\}. \tag{13}$$

Then there exists a decomposition of R^n into a direct sum of subspaces S_0, S, S_1 such that

$$N = \begin{pmatrix} N' & * & * \\ 0 & N_1 & * \\ 0 & 0 & N'' \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & 0 & I \\ 0 & H_1 & 0 \\ I & 0 & 0 \end{pmatrix}, \tag{14}$$

where

$$N' : \ S_0 \to S_0, \qquad N' = N_1' \oplus \ldots \oplus N_{p+q}',$$

$$N_j' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \qquad j = 1, \ldots p + q, \tag{15}$$

$$N'' : \ S_1 \to S_1, \qquad N'' = N_1'' \oplus \ldots \oplus N_{p+q}'',$$

$$N_j'' = N_j' \ \ if \ 1 \leq j \leq p, \ \ N_j'' = N_j'^{*} \ \ if \ p < j \leq p + q, \tag{16}$$

the internal operator N_1 is H_1-normal and the pair $\{N_1, H_1\}$ is determined up to unitary similarity. To go over from one decomposition $R^n = S_0 \dot{+} S \dot{+} S_1$ to another by means of a transformation T it is necessary that the matrix T be block triangular with respect to both decompositions.

According to this proposition, for an indecomposable operator N the subspace S_0 defined in (13) is neutral so that its dimension does not exceed k. Therefore, if we prove that for $n > 10[k/2] - 2$ the condition $dim\, S_0 \leq k$ fails, this will mean the decomposability of N.

According to [1, the proof of Lemma 1], if an H-normal operator N acting in C^n has the two distinct eigenvalues λ, $\overline{\lambda}$, then there exists a decomposition of C^n into a direct sum of subspaces V_1, V_2, V_3, V_4 such that

$$N = \begin{pmatrix} N|_{V_1} = N_1 & 0 & 0 & 0 \\ 0 & N|_{V_2} = N_2 & 0 & 0 \\ 0 & 0 & N|_{V_3} = N_3 & 0 \\ 0 & 0 & 0 & N|_{V_4} = N_4 \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & H_3 & 0 \\ 0 & 0 & 0 & H_4 \end{pmatrix}.$$

Here N_1 and N_3 have only one eigenvalue λ, N_2 and N_4 only one eigenvalue $\overline{\lambda}$, and $dim\, V_1 = dim\, V_2$. In our case C^n is R^n complexified, therefore, $dim\, V_3 = dim\, V_4$ too. Either V_1 or V_3 may be equal to zero.

Let $n > 10[k/2]-2$, i.e, $n \geq 10[k/2]$. Consider the following three cases: (a) $V_1 = V_2 = 0$, (b) $V_3 = V_4 = 0$, (c) $dim\, V_1 > 0$ and $dim V_3 > 0$.

(a) If $V_1 = 0$, then $dim\, V_3 (= dim\, V_4) \geq 5[k/2]$. Let H_3 (H_4) have $v_{-(3)}$ ($v_{-(4)}$) negative eigenvalues. Without loss of generality it can be assumed that $k = v_- = v_{-(3)} + v_{-(4)}$ so that $\min\{v_{-(3)}, v_{-(4)}\} \leq [k/2]$. Let $v_{-(3)} \leq [k/2]$. Decompose N_3 into an H-orthogonal sum of indecomposable operators $N_3^{(1)}$, $N_3^{(2)}$, ..., $N_3^{(m)}$: $N_3 = N_3^{(1)} \oplus N_3^{(2)} \oplus \ldots \oplus N_3^{(m)}$, $H_3 = H_3^{(1)} \oplus H_3^{(2)} \oplus \ldots \oplus H_3^{(m)}$, $V_3 = V_3^{(1)} \dot{+} V_3^{(2)} \dot{+} \ldots \dot{+} V_3^{(m)}$. Denote by $v_{-(3)}^{(j)}$ the number of negative eigenvalues of $H_3^{(j)}$ $(j = 1, \ldots, m)$. Let

$$V_3' = \sum_{v_{-(3)}^{(j)} > 0} V_3^{(j)}, \qquad V_3'' = \sum_{v_{-(3)}^{(j)} = 0} V_3^{(j)},$$

H_3', H_3'' and V_3', V_3'' be the corresponding sums of $H_3^{(j)}$ and $V_3^{(j)}$. Since for $v_{-(3)}^{(j)} > 0$ the condition $dim V_3^{(j)} \leq 4 v_{-(3)}^{(j)}$ holds [2, Theorem 1], we have $dim\, V_3' \leq 4 v_{-(3)} \leq 4[k/2]$. If $v_{-(3)}^{(j)} = 0$, then $dim\, V_3^{(j)} = 1$, $H_3^{(j)} = (1)$, and $N_3^{(j)} = (\lambda)$. Thus, $dim V_3'' \geq [k/2]$ and $Nz = \lambda z$, $N^{[*]} z = \overline{\lambda} z$ for all $z \in V_3''$. The operators N_3' and $N_3'^{[*]}$ commute so that if $dim\, V_3' > 0$, there exists at least one vector $z_0 \in V_3'$ such that $N z_0 = \lambda z_0$ and $N^{[*]} z_0 = \overline{\lambda} z_0$. If $dim\, V_3' = 0$, all nontrivial vectors from V_3 are eigenvectors of N corresponding to the eigenvalue λ and those of $N^{[*]}$ corresponding to the eigenvalue $\overline{\lambda}$. Therefore, in either case there exist at least $p = [k/2] + 1$ linearly independent vectors $\{z_l\}_{l=1}^p$ such that $N z_l = \lambda z_l$, $N^{[*]} z_l = \overline{\lambda} z_l$. Therefore, $dim S_0 \geq 2([k/2] + 1) > k$.

(b) If $dim V_3 = 0$, then $n = 2\, dim\, V_1$. Since no neutral subspace of a space of rank k can be of dimension more than k, $dim\, V_1 \leq k$ so that $n \leq 2k$. But it was proved before that $n \geq 2k$. Thus, in this case $n = 2k < 10[k/2] - 2$.

(c) If $dim V_1,\ dim V_3 > 0$, we can assume, as in the case (a) above, that $v_{-(3)} \leq [k/2] - dim V_1$ (the notation here is also as in (a)). Since $dim\, V_3 \geq 5[k/2] - dim\, V_1$, there are p linearly independent vectors $\{z_l\}_{l=1}^p$ such that $N z_l = \lambda z_l$, $N^{[*]} z_l = \overline{\lambda} z_l$ (p is equal to $[k/2] + 3 dim V_1 + 1$ if $dim V_3' > 0$ or to $5[k/2] - dim V_1$ if $dim V_3' = 0$). Since $dim V_1 \leq k$ and $k \geq 3$, we have $5[k/2] - dim V_1 \geq [k/2] + 1$. Thus, again $dim S_0 > k$ so that N is decomposable. The upper bound in (C) is proved. Step 1 is completed.

Step 2. Let us show the strictness of the low bound in (C) for even numbers k. Consider the pair of $2k \times 2k$ matrices

$$N = \begin{pmatrix} A & I_2 & 0 & \ldots & 0 & 0 \\ 0 & A & I_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & A & I_2 \\ 0 & 0 & 0 & \ldots & 0 & A \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & \ldots & 0 & I_2 \\ 0 & 0 & \ldots & I_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & I_2 & \ldots & 0 & 0 \\ I_2 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

where

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \qquad (\alpha, \beta \in \Re,\ \beta > 0) \tag{17}$$

(throughout what follows, by A we will denote the matrix (17)). It is seen that H has k negative and k positive eigenvalues. It can easily be checked by direct calculation that N is H-normal. The number of linearly independent vectors z_l satisfying the condition $N z_l = \lambda z_l$ ($\lambda = \alpha + i\beta$) is equal to 1, hence $dim S_0 = 2$. By [3, Proposition 3], if the subspace S_0 is two-dimensional, the operator N is indecomposable. So, the statement is proved and Step 2 is completed.

Step 3. For the case when k is odd consider the following pair of $2k \times 2k$ matrices

$$N = \begin{pmatrix} A & X & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & . & \vdots & \vdots & \vdots & . & \vdots & \vdots \\ 0 & 0 & \cdots & A & X & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & A & X & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & A^* & \cdots & 0 & 0 \\ \vdots & \vdots & . & \vdots & \vdots & \vdots & . & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & A^* & X \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & A^* \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & I_2 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I_2 & 0 \\ \vdots & \vdots & . & \vdots & \vdots & \vdots & . & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & I_2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & D_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & I_2 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & . & \vdots & \vdots & \vdots & . & \vdots & \vdots \\ 0 & I_2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ I_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

As $N^{[*]} = N$, the matrix N is H-normal. Since the condition $Nz = \lambda z$ ($\lambda = \alpha + i\beta$) implies

$$z = \begin{pmatrix} z_1 & iz_1 & 0 & \cdots & 0 \end{pmatrix}^T,$$

the subspace S_0 is two-dimensional and, according to [3, Proposition 3], the matrix N is indecomposable. Step 3 is completed.

Step 4. Consider the case (D). Let N have one real eigenvalue λ and two complex conjugate eigenvalues $\alpha \pm i\beta$. According to [3, Proposition 1], R^n is a direct sum of neutral subspaces $\mathcal{Q}_1$, $\mathcal{Q}_2$ such that $dim\mathcal{Q}_1 = dim\mathcal{Q}_2$, $N\mathcal{Q}_1 \subseteq \mathcal{Q}_1$, $N\mathcal{Q}_2 \subseteq \mathcal{Q}_2$, $N|_{\mathcal{Q}_1}$ has λ as its only eigenvalue, $N|_{\mathcal{Q}_2}$ has the two eigenvalues $\alpha \pm i\beta$. From the last condition it follows that $dim\mathcal{Q}_2$ is even. As in Step 1, case (b), we have $n = 2\,dim\mathcal{Q}_1 \leq 2k$, hence $n = 2k$ and $dim\mathcal{Q}_2 = k$ is necessarily an even number.

Now suppose that k is even and consider the following pair $\{N, H\}$:

$$N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix},$$

where the $k \times k$ submatrices N_1 and N_2 are as follows:

$$N_1 = \begin{pmatrix} A & I_2 & 0 & \ldots & 0 & 0 \\ 0 & A & I_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & A & I_2 \\ 0 & 0 & 0 & \ldots & 0 & A \end{pmatrix}, \qquad N_2 = \lambda I_k.$$

It is clear that the condition

$$N_1 N_2^* = N_2^* N_1 \tag{18}$$

is satisfied so that N is H-normal. Suppose that there exists a nondegenerate subspace V such that N is similar to the matrix

$$\widetilde{N} = \begin{pmatrix} N|_V = \widetilde{N_1} & 0 \\ 0 & N|_{V^{[\perp]}} = \widetilde{N_2} \end{pmatrix}. \tag{19}$$

Since the subspace generated by the eigenvectors corresponding to the eigenvalue $\alpha + i\beta$ is one-dimensional (in the complexified space), one of the submatrices $\widetilde{N_1}$ and $\widetilde{N_2}$ has λ as its only eigenvalue, therefore, either $\widetilde{N_1}$ or $\widetilde{N_2}$ is equal to λI. As in Theorem 1, Step 3, we conclude that under this condition V cannot be nondegenerate so that N is indecomposable. Step 4 is completed.

Step 5. Consider the case (E). That k is necessarily even and $n = 2k$ can be proved just as in Step 4 before. So, it remains to construct a pair $\{N, H\}$ satisfying the H-normality condition, where H has k negative and k positive eigenvalues, N is indecomposable, and N has only two pairs of complex conjugate eigenvalues $\alpha_1 \pm i\beta_1$, $\alpha_2 \pm i\beta_2$ ($\beta_1,\ \beta_2 > 0$, $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$). Let

$$N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix},$$

where the $k \times k$ submatrices N_1 and N_2 are as follows:

$$N_1 = \begin{pmatrix} A_1 & I_2 & 0 & \dots & 0 & 0 \\ 0 & A_1 & I_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A_1 & I_2 \\ 0 & 0 & 0 & \dots & 0 & A_1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} A_2 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_2 \end{pmatrix}.$$

Here

$$A_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ -\beta_2 & \alpha_2 \end{pmatrix}.$$

It can easily be checked that the H-normality condition (18) is satisfied. As in Step 4 before, the assumption that N is similar to (19) implies that either $\widetilde{N_1}$ or $\widetilde{N_2}$ (suppose $\widetilde{N_2}$) has two eigenvalues $\alpha_2 \pm i\beta_2$ only. Therefore, there are $m = \frac{dim \widetilde{N_2}}{2}$ complex linearly independent eigenvectors $\{z_j = x_j + iy_j\}_{j=1}^m$ of $\widetilde{N_2}$ corresponding to the eigenvalue $\alpha_2 + i\beta_2$. Consequently, the set $\{x_j\}_{j=1}^m \cup \{y_j\}_{j=1}^m$ is a basis of $\widetilde{N_2}$. But $[x_j, x_l] = [y_j, y_l] = [x_j, y_l] = 0$ for all $j, l = 1, \dots m$. Therefore, the subspace V cannot be nondegenerate and hence N is indecomposable. Step 5 is completed. The theorem is proved. $\square$

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